Orthogonal polynomials, special functions and mathematical physics

Miguel Lorente

Departamento de Física, Universidad de Oviedo, 33007 Oviedo, Spain

In the 6th Symposium of OPSA, there have been several communications dealing with concrete applications of orthogonal polynomials to experimental and theoretical physics, chemistry, biology and statistics. Here, I make suggestions concerning the use of the powerful apparatus of orthogonal polynomials and special functions in several lines of research in mathematical physics.

(1) *Hilbert spaces*: The postulates of quantum mechanics introduced an abstract space where the states of a system were represented by function-valued vectors and the physical magnitudes by operators. The formalism best suited to this model was the recently invented Hilbert space, where the basis was given by orthogonal polynomials [3]. The newly discovered orthogonal polynomials in the 20th century and the recently proposed multiple orthogonal polynomials open the door to new applications to more elaborate quantum models [14,16].

(2) *Integrable models*: A classical or quantum system can be described rigorously and exactly by some linear or nonlinear differential equation whose solution, except in a few cases, is not known. Very hard work has been done recently to integrate such a system with the help of inverse scattering methods, or soliton solutions and lattice approximations [10]. In some cases the solutions of the non-linear equation can be expressed in terms of solutions of classical special functions (Airy, Whittaker, Bessel, parabolic cylinder functions [5,6]). In the case of linear equations knowledge of the newly discovered orthogonal polynomials and the corresponding differential equation can be used to compare with the equation of the physical system in order to solve it.

(3) *Group representations and special functions*: The symmetries of some physical systems provide a powerful tool for calculating solutions [9]. To each symmetry there corresponds a group. The carrier space of some irreducible representation of a group can be expressed with the help of special functions [15]. In this symposium there have been proposed some particular groups associated to the Lagrangian, with the help of root systems of a classical Lie group the solution of which are given by the associated orthogonal polynomials. The class of quantum integrable systems associated with root systems was introduced by Olshanetsky and Perelomov as a generalization of the Calogero- Sutherland systems [12]. For the potential $v(q) = k(k-1)/(sinq)^2$ the wave functions of such a systems are related to polynomials in l variables (where l is the rank of the root systems) and they are a generalization of the Gegenbauer and Jack polynomials, that are connected to physical systems of l particles [13]. In this symposium similar arguments have been proposed for the Weyl group of type B_n; there are associated a Lagrangian and the spherical harmonics proposed by Dunkl. For the root system C_n there are associated the Macdonald-Koornwinder polynomials. These new methods open the way to constructing orthogonal polynomials of several variables and the corresponding recurrence relations and raising and lowering operators.

(4) *Orthogonal polynomials of a discrete variable*: Although the Kravchuk, Meixner, Charlier and Hahn polynomials of a discrete variable were introduced many years ago [8], the application to physical models on the lattice has started recently. First, the newly developed formalism of quantum groups, q-analysis, q-orthogonal polynomials and special functions can be considered as special cases of orthogonal polynomials on non-homogeneous lattices, and they are widely used in perturbative methods of quantum field theories [2]. Also the classical orthogonal polynomials and function of discrete variable have been applied to quantum systems such as the harmonic oscillator, the hydrogen atom and the Dirac equation on the lattice [4]. For some arbitrary one-dimensional systems, such as the Toda lattice, the evolution of the system in discrete time can be obtained exactly with the help of the orthogonal polynomial the argument of which are the symmetric product of the creation and annihilation operator of the harmonic oscillator. New insights into the solution of Dirac equation on the lattice have been obtained with the Wilson operators that satisfy a mild condition of chiral invariance [7]. Finally, in quantum gravity, a new area of research, the method of Ponzano-Regge calculus and the equivalent method of Penrose [11] spin-networks has been worked out with the help of the $3j - n$ symbols, that are connected with the Hahn polynomials of a discrete variable, and are suited for the discretization of space and time of General Relativity [1].

References

[1] J.C. Baez, An introduction to spin foam models of background free theory of quantum gravity, in: H. Gausteter et al. (Eds.), Lecture Notes in Physics, Vol. 543, Geometry and Quantum Physics, Schladming, 1999, Springer, Berlin, 2000, pp. 25-93.

[2] M. Creutz, Quarks, Gluons and Lattices, Cambridge University Press, Cambridge, 1983.

[3] M. Lorente, Quantum mechanics on discrete space and time, in: M. Ferrero, A. van der Merwe, New Developments in the Fundamental Problems in Quantum Physics, Kluwer Academic Publishers, Dordrecht, 1997.

[4] M. Lorente, Continuous vs. discrete models for the quantum harmonic oscillator and the hydrogen atom, Phys. Lett.A 285 (2001) 119-126.

[5] M. Manas, L.M. Alonso, From Ramond fermions to Lamé equations for orthogonal curvilinear coordinates, Phys.Lett. B 436 (1998) 316-322.

[6] M. Manas, L.M. Alonso, E. Medina, Dressing methods for geometric nets II: orthogonal and Egorov nets, J. Phys.A- Math. and Gen. 33 (2000) 7181-7206.

[7] I. Montvay, G. Muenster, Quantum Fields on a Lattice, Cambridge University Press, Cambridge, 1994.

[8] A.F. Nikiforov, S.K. Suslov, V.B. Uvarov, Classical Orthogonal Polynomials of a Discrete Variable, Springer, Berlin, 1991.

[9] P. Olver, Application of Lie Groups to Diferential Equations, Springer, Berlin, 1986.

[10] P. Olver, A Quasi-exactly Solvable Travel Guide, Group 21, World ScientiFc, Singapore, 1997.

[11] R. Penrose, Angular momentum: an approach to combinatorial space-time, in: T. Bastin (Ed.), Quantum Theory and Beyond, Cambridge University Press, Cambridge, 1971.

[12] A. Perelomov, Y. Zeldovich, Quantum Mechanics: Selected Topics, World Scientific, Singapore, 1998.

[13] A. Perelomov, Quantum integrable systems and Clebsch-Gordan Series. II, J. Phys. A- Math: and Gen. 32 (1999) 8563-8576.

[14] W. van Assche, Multiple orthogonal polynomials, Computational and Applied Mathematics, presented at this symposium.

[15] N.J. Vilenkin, Special Functions and the Theory of Group Representations, American Mathematical Society, Providence, RI, 1968.

[16] A. WNunsche, Hermite and Laguerre 2D polynomials and 2D functions with applications, Advanced special functions and integration methods, *Melfi*, 2000, Proc. Melfi Sch. Adv. Top. Math. Phys., Vol. 2, Aracne, Rome, 2001, pp. 157-197.